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The Classical Spin-$\frac{1}{2}$ Tachyon Field

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Abstract

We present the classical theory of a set of two spinor fields patterned closely after the Dirac theory in two-component form, but each field obeys a modified Klein-Gordon equation, in which the sign of m^2 has been changed. The solution contains parts with real and imaginary frequencies, that contribute differently to conserved quantities such as the charge and the energy-momentum vector. We also show how the minimal interaction with the electromagnetic field is obtained.

1. *Introduction*

Recently, there has been a renewed interest (Arons & Sudarshan, 1968; Bilaniuk *et al.*, 1962; Dhar & Sudarshan, 1968; Feinberg, 1967) in the possible existence of particles that move with speed greater than the speed of light, the so-called tachyons, within the context of classical and quantum relativistic theories. Their energy-momentum vector is spacelike, that is,†

$$p^2 = -m^2 \tag{1.1}$$

where m is a real constant. The relativistic quantum mechanics of a spinless tachyon could be based on the modified Klein-Gordon equation,

$$(\partial^2 - m^2)\,\varphi(x) = 0 \tag{1.2}$$

In this paper we present the analogous modification of the Dirac equation for a spin-$\frac{1}{2}$ tachyon. We obtain a conserved current density vector with a charge density that is *not* positive definite. We write down the general solution of the free-field equations in terms of the usual momentum-space expansion, and obtain the corresponding expressions for the charge and the energy-momentum vector. We find contributions both from real and imaginary frequencies k_0, with a peculiar dependence on the helicity of the former, and in the form of interference terms for the latter.

† We use the time-favoring metric (timelike vectors have positive norm) and the modified summation convention for repeated Greek lower indices that range from 0 to 3. Latin capitals are used for spinor indices. Other conventions are those used in Marx (1969, 1970).

In Section 2 we recast some of the usual results of the Dirac theory in terms of two-component spinors, which we find more appropriate for our purpose. The following section deals with the properties of the basic causal Green function for equation (1.2); we note that the imaginary-frequency components which have to be specified at the initial (final) time decay exponentially with increasing (decreasing) time. In Section 4 we present the theory of a spin-$\frac{1}{2}$ tachyon field in terms of two two-component spinors; we give the equations of motion with a minimal interaction with the electromagnetic field, and solve the free-field equations. We conclude with some remarks, in particular about the Lorentz covariance of the theory.

2. *The Dirac Field*

The usual theory of the spin-$\frac{1}{2}$ field is based on the Lagrangian density

$$\mathscr{L}_D = \tfrac{1}{2}i(\bar{\psi}\gamma_\mu \psi_{,\mu} - \bar{\psi}_{,\mu}\gamma_\mu \psi) - m\bar{\psi}\psi \tag{2.1}$$

which we can express in terms of two-component spinors (Marx, 1970) as

$$\begin{aligned}\mathscr{L}_D = \tfrac{1}{2}i(\chi_{\dot{A}}^* \sigma_\mu^{\dot{A}B} \chi_{B,\mu} - \chi_{\dot{A},\mu}^* \sigma_\mu^{\dot{A}B} \chi_B - \varphi^{\dot{A}} \sigma_{\mu\dot{A}B} \varphi_{,\mu}^{*B} \\ + \varphi_{,\mu}^{\dot{A}} \sigma_{\mu\dot{A}B} \varphi^{*B}) + m(\varphi^{\dot{A}} \chi_{\dot{A}}^* + \varphi^{*A} \chi_A)\end{aligned} \tag{2.2}$$

The general solution of the Dirac equation for the free field,

$$(-i\gamma . \partial + m)\psi(x) = 0 \tag{2.3}$$

is

$$\begin{aligned}\psi(x) = (2\pi)^{-3/2} \int d^3p\, (m/E)^{1/2} \\ \times \sum_\lambda \{b_\lambda(\mathbf{p})\, u_\lambda(\mathbf{p}) \exp(-ip.x) + d_\lambda^*(\mathbf{p})\, v_\lambda(\mathbf{p}) \exp(ip.x)\},\end{aligned} \tag{2.4}$$

where the index λ ranges over the helicity states and takes the values +1 and −1, u_λ and v_λ are the usual momentum-space bispinors, and

$$E = p_0 = +(\mathbf{p}^2 + m^2)^{1/2} \tag{2.5}$$

With our present choice (Marx, 1970) of the matrices γ_μ, the corresponding solution for $\chi_A(x)$ is

$$\begin{aligned}\chi_A(x) = (2\pi)^{-3/2} \int d^3p\, (2E)^{-1/2} \sum_\lambda \{(E + \lambda|\mathbf{p}|)^{1/2} b_\lambda(\mathbf{p})\, \chi_A{}^\lambda(\hat{p}) \times \\ \times \exp(-ip.x) - (E - \lambda|\mathbf{p}|)^{1/2} d_\lambda^*(\mathbf{p})\, \chi_A{}^\lambda(-\hat{p}) \exp(ip.x)\}\end{aligned} \tag{2.6}$$

Noether's theorem allows us to find the conserved densities from either of the forms of $\mathscr{L}_D$. For instance, the current density is

$$j_\mu = \bar{\psi}\gamma_\mu \psi = \chi_{\dot{A}}^* \sigma_\mu{}^{\dot{A}B} \chi_B + \varphi^{\dot{A}} \sigma_{\mu\dot{A}B} \varphi^{*B} \tag{2.7}$$

which gives the positive definite conserved charge

$$Q = \int d^3x\, \psi^\dagger \psi = \int d^3x\, (\chi^\dagger \chi + \varphi^\dagger \varphi) \tag{2.8a}$$

$$Q = \int d^3p \sum_\lambda (|b_\lambda|^2 + |d_\lambda|^2) \tag{2.8b}$$

The causal Green function for the Dirac field is

$$S_F(x) = (i\gamma.\partial + m)\Delta_F(x) \tag{2.9}$$

where

$$\Delta_F(x) = (2\pi)^{-4} \int d^4k \exp(-ik.x)(k^2 - m^2 + i\epsilon)^{-1} \tag{2.10}$$

we pointed out in an earlier paper (Marx, 1969) that its use requires the specification of the *positive-frequency* amplitudes b_λ at the *initial* time, and the *negative-frequency* amplitudes d_λ at the *final* time.

3. *The Causal Green Function for Tachyons*

The basic Green functions for tachyons obey the modified inhomogeneous Klein-Gordon equation

$$(\partial^2 - m^2)\Delta_i^T(x) = -\delta(x) \tag{3.1}$$

The causal Green function can be obtained (Dhar & Sudarshan, 1968) from equation (2.10) by changing the sign of m^2,

$$\Delta_F^T(x) = (2\pi)^{-4} \int d^4k \exp(-ik.x)(k^2 + m^2 + i\epsilon)^{-1} \tag{3.2}$$

As usual, we can do the k_0-integration by residues, and we obtain

$$\Delta_F^T(x) = \begin{cases} -\frac{1}{2}i(2\pi)^{-3} \int d^3k\, k_0^{-1} \exp(-ik.x), & t > 0 \\ -\frac{1}{2}i(2\pi)^{-3} \int d^3k\, k_0^{-1} \exp(ik.x), & t < 0 \end{cases} \tag{3.3}$$

where

$$k_0 = \begin{cases} +(\mathbf{k}^2 - m^2)^{1/2}, & |\mathbf{k}| > m \\ -i(m^2 - \mathbf{k}^2)^{1/2}, & |\mathbf{k}| < m \end{cases} \tag{3.4}$$

The position of the two poles in the complex k_0-plane is determined by the $i\epsilon$ in the denominator; as $|\mathbf{k}|$ varies from ∞ to m to zero, one pole moves above the negative real axis to the vicinity of the origin and up on the left side of the imaginary axis, while the other pole remains symmetric to this one through the origin. This way, the poles do not cross the path of integration. The contributions from imaginary frequency terms decrease exponentially for increasing $|t|$ both for positive and negative time, that is, away from the disturbance at $t = 0$. If we remember that positive-frequency terms propagate forward in time, and negative-frequency terms backward, this behavior appears reminiscent of the cutoff that occurs below certain frequencies in electromagnetic wave guides.

If we want to obtain the advanced and retarded Green functions, we presumably have to deform the path of integration when the poles are on the imaginary axis, for $|\mathbf{k}| < m$.

4. *The Spin-½ Tachyon Field*

We obtain a Lagrangian density for a tachyon field from equation (2.2) by a simple change of signs, and we base our theory on the real Lorentz scalar

$$\mathscr{L} = \tfrac{1}{2}i(\chi_{\dot{A}}^{*}\sigma_{\mu}^{\dot{A}B}\chi_{B,\mu} - \chi_{\dot{A},\mu}^{*}\sigma_{\mu}^{\dot{A}B}\chi_{B} + \varphi^{\dot{A}}\sigma_{\mu\dot{A}B}\varphi_{,\mu}^{*B} - \varphi_{,\mu}^{\dot{A}}\sigma_{\mu\dot{A}B}\varphi^{*B} + m(\varphi^{\dot{A}}\chi_{\dot{A}}^{*} + \varphi^{*A}\chi_{A}) \tag{4.1}$$

The equations of motion are

$$-i\sigma_{\mu\dot{A}B}\varphi_{,\mu}^{\dot{A}} + m\chi_{B} = 0 \tag{4.2}$$

$$i\sigma_{\mu}^{\dot{A}B}\chi_{B,\mu} + m\varphi^{\dot{A}} = 0 \tag{4.3}$$

and substitution of $\varphi^{\dot{A}}$ from equation (4.3) into equation (4.2) gives

$$(\partial^{2} - m^{2})\chi_{B}(x) = 0 \tag{4.4}$$

indicating that we are describing tachyon fields. We also obtain the current density vector†

$$j_{\mu} = \chi_{\dot{A}}^{*}\sigma_{\mu}^{\dot{A}B}\chi_{B} - \varphi^{\dot{A}}\sigma_{\mu\dot{A}B}\varphi^{*B} \tag{4.5}$$

the stress-energy tensor

$$T_{\mu\nu} = \tfrac{1}{2}i(\chi_{\dot{A}}^{*}\sigma_{\mu}^{\dot{A}B}\chi_{B,\nu} - \chi_{\dot{A},\nu}^{*}\sigma_{\mu}^{\dot{A}B}\chi_{B} + \varphi^{\dot{A}}\sigma_{\mu\dot{A}B}\varphi_{,\nu}^{*B} - \varphi_{,\nu}^{\dot{A}}\sigma_{\mu\dot{A}B}\varphi^{*B}) - \mathscr{L}g_{\mu\nu} \tag{4.6}$$

and the angular momentum density tensor

$$\begin{aligned} M_{\mu\nu\rho} = x_{\nu}T_{\mu\rho} - x_{\rho}T_{\mu\nu} + \tfrac{1}{2}i(&\chi_{\dot{A}}^{*}\sigma_{\mu}^{\dot{A}B}\mathscr{S}_{\nu\rho B}{}^{C}\chi_{C} \\ &- \chi^{\dot{C}*}\mathscr{S}_{\nu\rho\dot{A}}^{*}{}^{\dot{C}}\sigma_{\mu}^{\dot{A}B}\chi_{B} - \varphi^{\dot{A}}\sigma_{\mu\dot{A}B}\mathscr{S}_{\nu\rho C}{}^{B}\varphi^{*C} \\ &+ \varphi^{\dot{C}}\mathscr{S}_{\nu\rho\dot{C}}^{*}{}^{\dot{A}}\sigma_{\mu\dot{A}B}\varphi^{*B}) \end{aligned} \tag{4.7}$$

where

$$\mathscr{S}_{\nu\rho B}{}^{C} = \tfrac{1}{4}(\sigma_{\nu\dot{D}B}\sigma_{\rho}^{\dot{D}C} - \sigma_{\rho\dot{D}B}\sigma_{\nu}^{\dot{D}C}) = \tfrac{1}{2}(\sigma_{\nu\dot{D}B}\sigma_{\rho}^{\dot{D}C} - g_{\nu\rho}\delta_{B}{}^{C}) \tag{4.8}$$

The charge density is now the *difference* of two positive terms, but we cannot associate each spinor field with a particle of definite charge, since the equations of motion for the free fields show that it is not possible to have one of them vanish without the other being zero too.

We can now introduce electromagnetic interactions by the usual gauge invariant substitution

$$\partial_{\mu} \to D_{\mu} = \partial_{\mu} - ieA_{\mu} \tag{4.9}$$

to obtain†

$$\begin{aligned} \mathscr{L}' = \tfrac{1}{2}i[&\chi_{\dot{A}}^{*}\sigma_{\mu}^{\dot{A}B}D_{\mu}\chi_{B} - (D_{\mu}{}^{*}\chi_{\dot{A}}^{*})\sigma_{\mu}^{\dot{A}B}\chi_{B} + \varphi^{\dot{A}}\sigma_{\mu\dot{A}B}D_{\mu}{}^{*}\varphi^{*B} \\ &- (D_{\mu}\varphi^{\dot{A}})\sigma_{\mu\dot{A}B}\varphi^{*B}] + m(\varphi^{\dot{A}}\chi_{\dot{A}}^{*} + \varphi^{*A}\chi_{A}) = \mathscr{L} + eA_{\mu}j_{\mu} \end{aligned} \tag{4.10}$$

† The sign of the second term in j_{μ} is unchanged, of course, if we denote that spinor field by φ^{*A} instead of $\varphi^{\dot{A}}$. In that case the gauge transformation that leaves $\mathscr{L}$ invariant is $\chi_{A} \to \chi_{A}\exp(i\alpha)$, $\varphi^{A} \to \varphi^{A}\exp(-i\alpha)$. The same argument determines whether we should use D_{μ} or $D_{\mu}{}^{*}$ for a particular term in $\mathscr{L}$ to introduce the electromagnetic interactions.

The equations of motion become

$$i\sigma_{\mu}^{\dot{A}B} D_{\mu} \chi_B + m\varphi^{\dot{A}} = 0 \tag{4.11}$$

$$-i\sigma_{\mu \dot{A}B} D_{\mu} \varphi^{\dot{A}} + m\chi_B = 0 \tag{4.12}$$

and we note that the current density (4.5) is still conserved.

The general solution of the free-field equations can be written in a form similar to that in equation (2.6),

$$\chi_A(x) = (2\pi)^{-3/2} \int d^3k\,(2k_0)^{-1/2} \sum_{\lambda} \{(|\mathbf{k}| + \lambda k_0)^{1/2} b_\lambda(\mathbf{k})\, \chi_A{}^{\lambda}(\hat{k}) \exp(-ik.x) + (|\mathbf{k}| - \lambda k_0)^{1/2} d_\lambda{}^*(\mathbf{k})\, \chi_A{}^{\lambda}(-\hat{k}) \exp(ik.x)\} \tag{4.13}$$

where k_0 is given by equation (3.4), and we can take the square root with a positive real part for $k_0^{1/2}$ and $(|\mathbf{k}| \pm \lambda k_0)^{1/2}$. We find the corresponding expansion for $\varphi^{\dot{A}}$ from equation (4.3),

$$\varphi^{\dot{A}}(x) = (2\pi)^{-3/2} \int d^3k\,(2k_0)^{-1/2} \sum_{\lambda} \{(|\mathbf{k}| - \lambda k_0)^{1/2} b_\lambda(\mathbf{k})\, \chi^{-\lambda * \dot{A}}(\hat{k}) \times \exp(-ik.x) + (|\mathbf{k}| + \lambda k_0)^{1/2} d_\lambda{}^*(k)\, \chi^{-\lambda * \dot{A}}(-\hat{k}) \exp(ik.x)\} \tag{4.14}$$

and we calculate the charge, with special care to allow for imaginary values of k_0; we obtain

$$Q = \int_R d^3k \sum_{\lambda} \{\lambda |b_\lambda(\mathbf{k})|^2 - \lambda |d_\lambda(\mathbf{k})|^2\} + \int_{R'} d^3k \sum_{\lambda} \{i\lambda b_\lambda{}^*(\mathbf{k})\, d_\lambda{}^*(-\mathbf{k}) - i\lambda b_\lambda(\mathbf{k})\, d_\lambda(-\mathbf{k})\} \tag{4.15}$$

where the region R (R') is defined by $|\mathbf{k}| > m$ ($|\mathbf{k}| < m$). We note that the charge associated with the different amplitudes depends not only on the sign of the (real) frequency, but also on the helicity, while the contribution from imaginary frequencies does not have the usual form, but appears as interference terms. In the latter case, we can select linear combinations of the amplitudes, such as

$$\begin{aligned} b_\lambda(\mathbf{k}) &= 2^{-1/2}[\beta_\lambda(\mathbf{k}) - i\delta_\lambda(\mathbf{k})] \\ d_\lambda(-\mathbf{k}) &= 2^{-1/2}[i\beta_\lambda{}^*(\mathbf{k}) + \delta_\lambda{}^*(\mathbf{k})] \end{aligned} \tag{4.16}$$

so that (with no sum over λ)

$$ib_\lambda{}^*(\mathbf{k})\, d_\lambda{}^*(-\mathbf{k}) - ib_\lambda(\mathbf{k}) d_\lambda(-\mathbf{k}) = |\beta_\lambda(\mathbf{k})|^2 - |\delta_\lambda(\mathbf{k})|^2 \tag{4.17}$$

A similar calculation leads to the energy,

$$P_0 = \int_R d^3k\, k_0 \sum_{\lambda} \{\lambda |b_\lambda(\mathbf{k})|^2 + \lambda |d_\lambda(\mathbf{k})|^2\} + \int_{R'} d^3k\, ik_0 \sum_{\lambda} \{-\lambda b_\lambda{}^*(\mathbf{k})\, d_\lambda{}^*(-\mathbf{k}) - \lambda b_\lambda(\mathbf{k})\, d_\lambda(-\mathbf{k})\} \tag{4.18}$$

which is obviously not positive definite, and the momentum

$$\mathbf{P} = \int_R d^3k\,\mathbf{k} \sum_\lambda \{\lambda|b_\lambda(\mathbf{k})|^2 + \lambda|d_\lambda(\mathbf{k})|^2\} + \int_{R'} d^3k\,\mathbf{k} \sum_\lambda \{i\lambda b_\lambda{}^*(\mathbf{k})\, d_\lambda{}^*(-\mathbf{k}) - i\lambda b_\lambda(\mathbf{k})\, d_\lambda(-\mathbf{k})\} \tag{4.19}$$

Equations (4.15), (4.18) and (4.19) (there also is a similar expression for the angular momentum tensor) provide the basis for the interpretation of the b_λ and d_λ as probability amplitudes in a relativistic quantum mechanics. They become creation or annihilation operators for the quantized field.

5. *Concluding Remarks*

We have presented the classical theory of a free spin-$\frac{1}{2}$ tachyon field, which can also interact with the electromagnetic field in the usual manner. The charge (electric or other) is no longer positive definite, as is the case in equations (2.8a) and (2.8b), and this could allow for the probabilistic interpretation (Marx, 1969) of quantum mechanics.

Since k_μ is not really the energy-momentum vector of a particle, but just arises from the Fourier decomposition of a field, we have kept all values of $\mathbf{k}$, while noticing the marked differences in the nature of the contributions to the field and conserved quantities from components with $|\mathbf{k}| > m$ and $|\mathbf{k}| < m$.

The Lagrangian density is a Lorentz scalar and leads to covariant equations of motion. Our solution of the free-field equations is not expressed in a covariant form, since we have used $\mathbf{x}$ and $\mathbf{k}$ as continuous indices, while t is the dynamical parameter and k_0 is a function of $\mathbf{k}$. This is a reflection of our choice of the k_0 variable to do the first integration for the Green function; the initial and final values have to be given on hyperplanes of constant t. In the case of imaginary-frequency components, the exponential behavior in time implies that the field diverges on any hyperplane normal to a timelike unit four-vector n not along the time axis, since it contains points for which $t = \pm\infty$. It is also possible to choose this vector n to represent the state of motion of the observer (Marx, 1970), which makes the expansion covariant but dependent on the observer; the exponential behavior then appears in the direction of n.

The real-frequency components behave more or less in the expected way, but an interpretation in terms of quantum mechanics still has to take into account the changes of sign of k_0 under orthochronous Lorentz transformations, and the dependence of the charge and energy-momentum on the helicity.

While it is clear that components with real and imaginary frequencies do not mix for a free field, we do not expect this to remain so when electromagnetic or other interactions are included.

No tachyons have been found to date; if they exist, it appears that their inclusion in the present framework of physics would be difficult, but not impossible.

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